

Remarks on Localizing Futaki-Morita Integrals At Isolated Degenerate Zeros

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Abstract

In this note we study the localization of Futaki-Morita integrals at isolated degenerate zeros by giving a streamlined exposition in the spirit of Bott [4] and implement the localization procedure for a holomorphic vector field on $\mathbb{C}P^n$ with a maximally degenerate zero, giving an essentially unique formula for the Futaki-Morita integral invariants without using a summation over multiple points. In a coming paper we will apply similar calculations to the Calabi-Futaki invariant of a Kähler blowup.

1 Introduction

Let M be an n -dimensional compact complex manifold and \mathfrak{h} the Lie algebra of holomorphic vector fields on M . An isolated zero p of $X \in \mathfrak{h}$ is called *nondegenerate* if for local coordinates (z_1, \dots, z_n) centered at p ,

$$X = \sum_{i,j} [a_{ij}z_i + O(z^2)] \frac{\partial}{\partial z_j}$$

the matrix $DX = (a_{ij})$ is invertible at p , i.e. $\det DX_p \neq 0$, and *degenerate* otherwise. Given a Hermitian metric on M , let Θ be the curvature of its Chern connection ∇ . The holomorphic localization theorem of Bott [4] (see also [13]) states:

Theorem 1.1 (*Bott [4]*) *Suppose $X \in \mathfrak{h}$ is such that $\text{Zero}(X)$ consists of isolated nondegenerate zeros $\{p_i\}$. For any invariant polynomial ϕ of degree n ,*

$$\int_M \phi \left(\frac{\sqrt{-1}}{2\pi} \Theta \right) = \sum_i \frac{\phi(DX_{p_i})}{\det DX_{p_i}} \quad (1)$$

Bott [3] extended this result to vector fields with positive dimensional but still nondegenerate zero locus (nondegenerate in the sense that DX is invertible in the normal direction to the zero locus).

When $\deg(\phi) < n$ the lefthand side of (1) is of course zero for dimensional reasons. A generalization to $\deg(\phi) > n$ was given by Futaki and Morita [12]: Let

$$E = \mathcal{L}_X - \nabla_X \quad (2)$$

where \mathcal{L}_X is the Lie derivative with respect to X . It is straightforward to check E defines a smooth endomorphism $E \in \Gamma(\text{End}(TM'))$ of the holomorphic tangent bundle TM' . The *Futaki-Morita* integral is

$$f_\phi(X) = \int_M \underbrace{\tilde{\phi}(E, \dots, E)}_{k \text{ copies}}, \frac{\sqrt{-1}}{2\pi} \Theta, \dots, \frac{\sqrt{-1}}{2\pi} \Theta$$

where $\tilde{\phi}$ is the polarization of an invariant polynomial ϕ of degree $n + k$. Futaki and Morita showed $f_\phi : \mathfrak{h} \rightarrow \mathbb{C}$ does not depend on the choice of metric (in Bott's theorem this follows from Chern-Weil theory) and by the same transgression argument used by Bott to prove Theorem 1.1 showed:

Theorem 1.2 (*Futaki-Morita [12]*) *Suppose that $X \in \mathfrak{h}$ has isolated, nondegenerate zeros $\{p_i\}$ and $E \in \Gamma(\text{End}(TM'))$ as in (2). Then*

$$\binom{n+k}{n} f_\phi(X) = (-1)^k \sum_i \frac{\phi(DX_{p_i})}{\det DX_{p_i}} \quad (3)$$

Futaki-Morita moreover showed that Futaki's invariant obstructing the existence of Kähler-Einstein metrics on compact Kähler manifolds with $c_1(M) > 0$ can be understood within this integral invariant framework (see section 2.4).

The proof of (3) is based on exhibiting the Futaki-Morita integral as a certain Grothendieck residue via transgression, and the Bochner-Martinelli kernel provides an explicit representative for the Grothendieck residue. Using properties of the Grothendieck residue and inserting a power series expansion into the transgression argument, we will show the following extension to the case of isolated degenerate zeros:

Theorem 1.3 *If the zero locus of $X \in \mathfrak{h}$ is a single isolated degenerate zero p such that in local coordinates centered at p*

$$z_i^{\alpha_i+1} = \sum b_{ij} X_j$$

for some matrix $B = (b_{ij})$ of holomorphic functions, then

$$\binom{n+k}{n} f_\phi(X) = (-1)^k \frac{1}{\prod \alpha_i!} \cdot \left. \frac{\partial^{|\alpha|} (\phi(DX) \det B)}{\partial z_1^{\alpha_1} \cdots \partial z_n^{\alpha_n}} \right|_{z=0} \quad (4)$$

If $\text{Zero}(X)$ consists of multiple isolated, possibly degenerate points then the Futaki-Morita integral is a sum over local contributions (4).

The existence of such an α is guaranteed by the strong Hilbert Nullstellensatz for analytic functions. In the case that X has nondegenerate zeros, one may take $B = DX^{-1}$ with $\alpha_i = 0$, and (3) is immediately recovered.

Theorem 1.3 follows from a simple power series expansion in Bott's transgression argument and application of well-known properties of Grothendieck residues. Surprisingly it does not seem to have received use in the literature although it has certainly been pointed out in related contexts [2] [15] [5] [7]. We give a complete presentation, hopefully contributing to the available exposition on Bott-style localization. The calculations in the last section serve to illustrate localization at a degenerate zero, even if the results are standard. We remark that Proposition 4.1 is essentially unique in that any vector field with a maximally degenerate zero on $\mathbb{C}P^n$ is equivalent to the one used, and thus any formula for Futaki-Morita invariants on $\mathbb{C}P^n$ not involving a summation over fixed points will be of the form arrived at.

One application of localization at degenerate zeros is to calculations on blow-ups: If X is a holomorphic vector field with nondegenerate zero at p , the blowup $\text{Bl}_p(M)$ at p admits a holomorphic lift \tilde{X} of X . Zeros of \tilde{X} in the exceptional divisor may very well be degenerate, depending on the linearization of X at p . We will study this in a forthcoming paper, in particular extending results of Li and Shi concerning the Futaki invariant of Kähler surface blow-ups [14]. The calculations used will be extensions of that in Proposition 4.1.

The paper is organized as follows: In section 2 we recall background material on invariant polynomials, Grothendieck residues, the Bochner-Martinelli kernel, and the Futaki invariant for clarity. In section 3 we give a complete proof of the main theorem, which may in particular be read as a self-contained proof of the results of Bott and Futaki-Morita. In section 4 we give our main calculation.

2 Background

2.1 Invariant Polynomials

Let $\mathfrak{gl}(n, \mathbb{C})$ denote the spaces of $n \times n$ matrices over \mathbb{C} . An *invariant polynomial* $\phi : \mathfrak{gl}(n, \mathbb{C}) \rightarrow \mathbb{C}$ is a homogeneous polynomial in the entries of $\mathfrak{gl}(n, \mathbb{C})$ such that $\phi(A) = \phi(gAg^{-1})$ for all $g \in GL(n, \mathbb{C})$.

We will consider two sources of input for an invariant polynomial ϕ :

1. Let $X \in \mathfrak{h}$ be a holomorphic vector field vanishing at p and consider $A = DX$. As coordinate change about p has the effect of conjugating DX , $\phi(DX)$ is locally a well-defined holomorphic function.
2. Let $E \in \Omega^k(\text{End}(TM'))$. Locally E is a k -form valued matrix that transforms according to $E_\alpha = g_{\alpha\beta} E_\beta g_{\alpha\beta}^{-1}$, where $g_{\alpha\beta}$ are the usual transition functions for

TM' . By the invariance hypothesis, $\phi(E) \in \Omega^*(M, \mathbb{C})$ given by point-wise evaluation in local coordinates is well-defined.

2.2 Grothendieck Residues

Let U be an open ball about the origin in \mathbb{C}^n and consider holomorphic functions $f_1, \dots, f_n \in \mathcal{O}(\overline{U})$ such that the origin is an isolated zero of $f = (f_1, \dots, f_n)$. The *Grothendieck residue* of

$$\omega = \frac{h(z) dz^1 \wedge \dots \wedge dz^n}{f_1(z) \cdots f_n(z)} \quad h \in \mathcal{O}(\overline{U})$$

at 0 is defined to be

$$\text{Res}_0(\omega) = \left(\frac{1}{2\pi\sqrt{-1}} \right)^n \int_{\Gamma} \omega. \quad (5)$$

where Γ is the real n -cycle $\Gamma = \{z \mid |f_i(z)| = \epsilon_i\}$, oriented by

$$d(\arg(f_1)) \wedge \dots \wedge d(\arg(f_n)) > 0.$$

Linearity of the residue is immediate, as is the fact that $\text{Res}_0 \omega$ depends only on the homology class $\Gamma \in H_n(U - D, \mathbb{Z})$ and cohomology class $[\omega] \in H_{DR}^n(U - D)$ where $D_i = f_i^{-1}(0)$ and $D = \bigcup D_i$.

An alternate description of the Grothendieck residue that employs a degree $2n - 1$ de Rham class is as follows: Let $U_i = U - D_i$ and consider the open cover $\{U_i\}$ of $U^* = U - \{0\}$. The meromorphic form ω can be thought of as a Čech $(n - 1)$ -coycle for the sheaf of holomorphic forms on U^* , which is trivially closed as there are only n open sets in the cover. We denote by η_ω the image of $\left(\frac{1}{2\pi\sqrt{-1}} \right)^n \omega$ under the Dolbeault isomorphism $\check{H}^{n-1}(U^*, \Omega^n) \cong H^{n,n-1}(U^*)$, and since $d = \bar{\partial}$ on forms of type $(n, n - 1)$ we may think of η_ω as an element of $H_{DR}^{2n-1}(U^*) \cong \mathbb{C}$. Here we are using that U^* has the homotopy type of the $2n - 1$ sphere.

It then turns out the Grothendieck residue is precisely the image of the following sequence of maps:

$$\text{Res}_0 : \left(\frac{1}{2\pi\sqrt{-1}} \right)^n \omega \longmapsto \eta_\omega \longmapsto \int_{S^{2n-1}} \eta_\omega \in \mathbb{C} \quad (6)$$

We refer to Griffiths and Harris [13] for the calculation.

Lemma 2.1 (Transformation Rule) *Suppose that $g = (g_1, \dots, g_n)$ satisfies the same hypotheses as f above and moreover that*

$$g_i(z) = \sum_j b_{ij}(z) f_j(z),$$

for some matrix $B(z) = (b_{ij}(z))$ of holomorphic functions. Then for any $h(z) \in \mathcal{O}(\overline{U})$,

$$\text{Res}_0 \frac{h(z) dz^1 \wedge \cdots \wedge dz^n}{f_1(z) \cdots f_n(z)} = \text{Res}_0 \frac{h(z) \det B(z) dz^1 \wedge \cdots \wedge dz^n}{g_1(z) \cdots g_n(z)}$$

We refer to p. 657-659 of [13] for a full proof. The key idea is the notion of a *good deformation* of f , namely a family $f_t = (f_{1,t}, \dots, f_{n,t})$ of holomorphic functions on U satisfying the same hypotheses as f_i , continuous in t , with $f_0 = f$, and such that for $t > 0$ the Jacobian of f_t is invertible. A Sard's Theorem argument proves the existence of such a good deformation. The lemma follows by establishing the transformation law in the case of an invertible Jacobian and taking an appropriate limit as $t \rightarrow 0$.

2.3 Bochner-Martinelli Formula

The *Bochner-Martinelli* kernel is defined on $\mathbb{C}^n \times \mathbb{C}^n$ by

$$\beta(w, z) = C_n \sum_{i=1}^n \frac{(\overline{w}_i - \overline{z}_i) \overline{\Phi_i(w - z)} \wedge \Phi(w)}{\|w - z\|^{2n}}$$

where

$$\begin{aligned} C_n &= (-1)^{n(n-1)/2} \frac{(n-1)!}{(2\pi\sqrt{-1})^n} \\ \Phi_i(w) &= (-1)^{i-1} dw^1 \wedge \cdots \wedge \widehat{dw^i} \wedge \cdots \wedge dw^n \\ \Phi(w) &= dw^1 \wedge \cdots \wedge dw^n \end{aligned}$$

Key properties are:

1. $\bar{\partial}\beta(w, z) = 0$ as a function of w away from the diagonal $w = z$.
2. The constant C_n is such that $\int_{\partial B_\epsilon(0)} \beta(w, 0) = 1$, where $B_\epsilon(0)$ is any ball in \mathbb{C}^n centered at 0 and integration is with respect to w .

The Bochner-Martinelli kernel may be used to construct an explicit representative of the class η_ω in (6): Given f, ω, η_ω as before, let $F : U \rightarrow \mathbb{C}^n \times \mathbb{C}^n$ be $F(z) = (z + f(z), z)$. It follows that

$$\eta_\omega = h(z) F^* \beta(w, z)$$

is a distinguished representative of the class $\left[\left(\frac{1}{2\pi\sqrt{-1}} \right)^n \omega \right]$. In other words,

$$\text{Res}_0(\omega) = C_n \int_{\partial B_\epsilon(0)} h(z) \sum_{i=1}^n \frac{(-1)^{i-1} \bar{f}_i d\bar{f}_1 \wedge \cdots \wedge \widehat{d\bar{f}_i} \wedge \cdots \wedge d\bar{f}_n \wedge dz_1 \wedge \cdots \wedge dz_n}{\|f\|^{2n}} \quad (7)$$

2.4 Futaki Invariant

Let M be an n -dimensional compact Kähler manifold. Establishing the existence of various canonical metrics on M is one of the central problems in Kähler geometry. See [16] for a survey. In the search for Kähler-Einstein metrics, the first Chern class $c_1(M)$ is necessarily definite or zero according to the sign of the Ricci curvature, imposing a strong topological restriction. The celebrated works of Yau [18] and Aubin, Yau [1] [18] settled existence and uniqueness of Kähler-Einstein metrics in the cases of $c_1(M) = 0$ and $c_1(M) < 0$, respectively. When M has positive first Chern class there are well-known obstructions and the problem has only recently been settled in the work of Chen-Donaldson-Sun [6]; see also Tian [17].

We recall Futaki's obstruction to Kähler-Einstein metrics when $c_1(M) > 0$. Choose a Kähler metric $\omega \in 2\pi c_1(M)$. Since $\text{Ric}(\omega) \in 2\pi c_1(M)$ as well, by the $\partial\bar{\partial}$ -lemma

$$\text{Ric}(\omega) - \omega = \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial} F_\omega$$

for some real-valued function F_ω (defined up to addition of a constant). The metric ω is called *Kähler-Einstein* if F_ω is constant.

Futaki [9] [10] defined what is now called the *Futaki invariant*

$$\text{Fut}(X, \omega) = \int_M X(F_\omega) \omega^n$$

and showed the definition does not depend on the choice of ω within its Kähler class. The vanishing of $\text{Fut}(X, \omega)$ is thus necessary for the existence of a Kähler-Einstein metric.

Futaki and Morita [11] [12] showed that the Futaki invariant may be understood within the Futaki-Morita integral invariant framework. Specifically, they proved

$$\text{Fut}(X, \omega) = f_\phi(X) \tag{8}$$

where ϕ is the invariant polynomial $\phi(A) = \text{Tr}(A^{n+1})$. By (3), when X has isolated nondegenerate zeros $\{p_i\}$,

$$\text{Fut}(X, \omega) = -\frac{1}{n+1} \sum_i \frac{\text{Tr}(DX_{p_i})^{n+1}}{\det DX_{p_i}}$$

3 Proof of Theorem 1.3

We now turn to the proof of Theorem 1.3, which will use:

Lemma 3.1 *Suppose $f = (f_1, \dots, f_n)$ is holomorphic and has an isolated zero at $z = 0$, and let $B = (B_{ij})$ be a matrix of holomorphic functions such that*

$$z_i^{\alpha_i+1} = \sum_{j=1}^n B_{ij} f_j$$

for some $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}_{\geq 0}^n$. Then

$$\text{Res}_0 \left[\frac{h(z) dz^1 \wedge \dots \wedge dz^n}{f_1 \dots f_n} \right] = \frac{1}{\prod \alpha_i!} \cdot \frac{\partial^{|\alpha|} (h(z) \det B)}{\partial z_1^{\alpha_1} \dots \partial z_n^{\alpha_n}} \Big|_{z=0}$$

where $|\alpha| = \sum \alpha_i$.

Proof Since Lemma 2.1 holds for possibly singular B ,

$$\text{Res}_0 \left[\frac{h(z) dz^1 \wedge \dots \wedge dz^n}{f_1 \dots f_n} \right] = \text{Res}_0 \left[\frac{h(z) \det B dz^1 \wedge \dots \wedge dz^n}{z_1^{\alpha_1+1} \dots z_n^{\alpha_n+1}} \right].$$

Expand the holomorphic function $h(z) \det B$ in a neighborhood of $z = 0$:

$$h(z) \det B = \sum_{\gamma \geq 0} \frac{1}{\gamma!} \cdot \frac{\partial^{|\gamma|} (h(z) \det B)}{\partial z_1^{\gamma_1} \dots \partial z_n^{\gamma_n}} \Big|_{z=0} z_1^{\gamma_1} \dots z_n^{\gamma_n}$$

By linearity of the Grothendieck residue

$$\text{Res}_0 \left[\frac{h(z) dz^1 \wedge \dots \wedge dz^n}{f_1 \dots f_n} \right] = \sum_{\gamma \geq 0} \frac{1}{\gamma!} \cdot \frac{\partial^{|\gamma|} (h(z) \det B)}{\partial z_1^{\gamma_1} \dots \partial z_n^{\gamma_n}} \Big|_{z=0} \text{Res}_0 \left[\frac{dz^1 \wedge \dots \wedge dz^n}{z_1^{\alpha_1-\gamma_1+1} \dots z_n^{\alpha_n-\gamma_n+1}} \right]$$

The lemma then follows from the definition of Grothendieck residue and the multidimensional Cauchy integral formula, which shows all terms with $\gamma_i \neq \alpha_i$ vanish while terms with $\gamma_i = \alpha_i$ produce a residue of 1. \blacksquare

Let us define forms

$$\phi_r = \binom{n+k}{r} \tilde{\phi}(\underbrace{E, \dots, E}_{n+k-r \text{ times}}, \underbrace{\Theta, \dots, \Theta}_{r \text{ times}}) \in \Omega^{r,r}(M, \mathbb{C})$$

where $\tilde{\phi}$ is the polarization of invariant polynomial ϕ of degree $n+k$ and E as in (2). It is ϕ_n in which we are ultimately interested for dimensional reasons.

Also let $\hat{M} = M - \bigcup B_\epsilon(p_i)$ where $B_\epsilon(p_i)$ denotes small disjoint balls about the $p_i \in \text{Zero}(X)$. Upon choice of a Hermitian metric g on M , define

$$\eta(\cdot) = \frac{g(\cdot, \bar{X})}{\|X\|^2} \in \Omega^{1,0}(\hat{M})$$

$$\Phi_i = \eta \wedge \phi_i \wedge (\bar{\partial}\eta)^{n-i-1} \in \Omega^{n,n-1}(\hat{M})$$

$$\Phi = \sum_{i=0}^{n-1} \Phi_i = \eta \wedge \sum_{i=0}^{n-1} \phi_i \wedge (\bar{\partial}\eta)^{n-i-1} \in \Omega^{n,n-1}(\hat{M})$$

Lemma 3.2 *With the above definitions,*

1. $\bar{\partial}\phi_i = i_X \phi_{i+1}$ for $i = 0, \dots, n-1$
2. $i_X \bar{\partial}\eta = 0$
3. $i_X \bar{\partial}\Phi_i = i_X \phi_i \wedge (\bar{\partial}\eta)^{n-i} - i_X \phi_{i+1} \wedge (\bar{\partial}\eta)^{n-i-1}$ for $i = 0, \dots, n-1$

As a result, on \hat{M} :

$$\bar{\partial}\Phi + \phi_n = 0 \tag{9}$$

Proof We first show

$$\bar{\partial}E = i_X \Theta \tag{10}$$

As \mathcal{L}_X preserves the type of a form when X is holomorphic, and $\mathcal{L}_X = i_X d + di_X$ by Cartan's formula, it follows from the decomposition $d = \partial + \bar{\partial}$ that

$$i_X \bar{\partial} + \bar{\partial}i_X = 0 \tag{11}$$

Equation (10) follows by computing $\bar{\partial}E$ applied to a local holomorphic section σ of TM' :

$$\begin{aligned} \bar{\partial}(E\sigma) &= \bar{\partial}(\mathcal{L}_X \sigma - i_X \nabla \sigma) \\ &= 0 + i_X \bar{\partial} \nabla \sigma \quad (\text{using (11)}) \\ &= i_X \Theta \sigma \end{aligned}$$

- 1.) Using the symmetry of $\tilde{\phi}$, equation (10), and that $\bar{\partial}\Theta = 0$,

$$\begin{aligned} \bar{\partial}\phi_i &= \binom{n}{i} \bar{\partial}\tilde{\phi}(E, \dots, E, \overbrace{\Theta, \dots, \Theta}^{i \text{ times}}) \\ &= \binom{n}{i} (n-i) \tilde{\phi}(E, \dots, E, \bar{\partial}E, \Theta, \dots, \Theta) \\ &= \binom{n}{i} (n-i) \tilde{\phi}(E, \dots, E, i_X \Theta, \Theta, \dots, \Theta) \\ &= \binom{n}{i} \frac{(n-i)}{(i+1)} i_X \tilde{\phi}(E, \dots, E, \overbrace{\Theta, \dots, \Theta}^{i+1 \text{ times}}) \\ &= i_X \phi_{i+1} \end{aligned}$$

2.) Since $i_X \eta = 1$,

$$0 = \bar{\partial}(i_X \eta) = -i_X(\bar{\partial}\eta).$$

We are again using $i_X \bar{\partial} = -\bar{\partial} i_X$ as in (11).

3.) By the first two parts of the lemma and $i_X \eta = 1$,

$$\begin{aligned} i_X \bar{\partial} \Phi_i &= i_X [\phi_i \wedge (\bar{\partial}\eta)^{n-i} - \eta \wedge \bar{\partial}\phi_i \wedge (\bar{\partial}\eta)^{n-i-1}] \\ &= i_X [\phi_i \wedge (\bar{\partial}\eta)^{n-i} - \eta \wedge i_X \phi_{i+1} \wedge (\bar{\partial}\eta)^{n-i-1}] \\ &= i_X \phi_i \wedge (\bar{\partial}\eta)^{n-i} - i_X \eta \wedge i_X \phi_{i+1} \wedge (\bar{\partial}\eta)^{n-i-1} \\ &= i_X \phi_i \wedge (\bar{\partial}\eta)^{n-i} - i_X \phi_{i+1} \wedge (\bar{\partial}\eta)^{n-i-1} \end{aligned}$$

We now prove (9):

$$\begin{aligned} i_X \bar{\partial} \Phi &= i_X \bar{\partial} \sum_{i=0}^{n-1} \Phi_i \\ &= \sum_{i=0}^{n-1} i_X \phi_i \wedge (\bar{\partial}\eta)^{n-i} - i_X \phi_{i+1} \wedge (\bar{\partial}\eta)^{n-i-1} \\ &= i_X \phi_0 \wedge (\bar{\partial}\eta)^n - i_X \phi_n \\ &= -i_X \phi_n \end{aligned}$$

where we have used that $i_X \phi_0$ is trivially 0. Thus $i_X(\bar{\partial}\Phi + \phi_n(\Theta)) = 0$ on \hat{M} and so

$$\bar{\partial}\Phi + \phi_n = 0$$

since i_X is injective on top degree forms away from $\text{Zero}(X)$. ■

With these preliminaries out of the way, we are ready to prove Theorem 1.3. The transgression formula (9) reduces calculation to a neighborhood of $\text{Zero}(X)$:

$$\begin{aligned} \int_M \phi_n &= \lim_{\epsilon \rightarrow 0} \int_{\hat{M}} \phi_n \\ &= - \lim_{\epsilon \rightarrow 0} \int_{\hat{M}_\epsilon} \bar{\partial}\Phi && \text{(by (9))} \\ &= - \lim_{\epsilon \rightarrow 0} \int_{\hat{M}_\epsilon} d\Phi && \text{(since } \Phi \text{ is type } (n, n-1)) \\ &= - \lim_{\epsilon \rightarrow 0} \sum_i \int_{\partial B_\epsilon(p_i)} \Phi && \text{(by Stokes' Theorem)} \end{aligned} \tag{12}$$

These local contributions will be computed using a Hermitian metric g that is Euclidean on a neighborhood of each p_i (although the form Φ depends on the choice of g , by Futaki and Morita's work $f_\Phi(X)$ does not). To be precise, consider the open cover of M by disjoint $U_i = B_\epsilon(p_i)$ and $U_0 = M - \cup \overline{B_{\epsilon/2}(p_i)}$. Let $\{\rho_i\}$ be a partition of unity subordinate to this cover and g_i be the Euclidean metric on U_i for $i \neq 0$, and let g_0 be any Hermitian metric on U_0 . Then $g = \sum \rho_i g_i$ is the Hermitian metric on M we work with.

In the Euclidean metric, $\eta = \frac{\sum \overline{X^i} dz^i}{\|X\|^2}$ so that

$$\bar{\partial}\eta = \frac{\sum d\overline{X^i} \wedge dz^i}{\|X\|^2} - \frac{\sum \overline{X^i} X^j d\overline{X^j} \wedge dz^i}{\|X\|^4}$$

Notice that the second term of $\bar{\partial}\eta$ wedged with itself is zero by symmetry, as it is when wedged with η . We therefore find by direct computation

$$\eta \wedge (\bar{\partial}\eta)^{n-1} = -(-1)^{n(n-1)/2} (n-1)! \sum_i \frac{(-1)^{i-1} \overline{X^i} d\overline{X^1} \wedge \dots \wedge \widehat{d\overline{X^i}} \wedge \dots \wedge d\overline{X^n} \wedge dz^1 \wedge \dots \wedge dz^n}{\|X\|^{2n}}$$

In terms of the Grothendieck residue (7), for any holomorphic h we have

$$\left(\frac{\sqrt{-1}}{2\pi}\right)^n \int_{\partial B_{\epsilon/2}(p)} h(z) \eta \wedge (\bar{\partial}\eta)^{n-1} = (-1)^{n+1} \text{Res}_p \left[\frac{h(z) dz^1 \wedge \dots \wedge dz^n}{X^1 \dots X^n} \right] \quad (13)$$

Since g is Euclidean near $p \in \text{Zero}(X)$, $\Gamma_{ij}^k = 0$ and so

$$E|_{B_{\epsilon/2}(p)} = -\frac{\partial X^j}{\partial z^k} \frac{\partial}{\partial z^j} \otimes dz^k$$

It follows $\phi(E)|_{B_{\epsilon/2}(p)} = (-1)^{n+k} \phi(DX)$. And as $\Theta = 0$ near p as well,

$$\Phi|_{B_{\epsilon/2}(p)} = \eta \wedge \phi_0 \wedge (\bar{\partial}\eta)^{n-1} = (-1)^{n+k} \phi(DX) \eta \wedge (\bar{\partial}\eta)^{n-1} \quad (14)$$

We finish the proof by continuing the above calculation with these observations,

$$\begin{aligned}
\binom{n+k}{n} f_\phi(X) &= \int_M \left(\frac{\sqrt{-1}}{2\pi} \right)^n \phi_n \\
&= -\lim_{\epsilon \rightarrow 0} \sum_i \int_{\partial B_{\epsilon/2}(p_i)} \left(\frac{\sqrt{-1}}{2\pi} \right)^n \Phi && \text{(by 12)} \\
&= -\lim_{\epsilon \rightarrow 0} \sum_i \left(\frac{\sqrt{-1}}{2\pi} \right)^n \int_{\partial B_{\epsilon/2}(p_i)} (-1)^{n+k} \phi(DX) \eta \wedge (\bar{\partial} \eta)^{n-1} && \text{(by (14))} \\
&= -(-1)^{n+k} \sum_i (-1)^{n+1} \text{Res}_{p_i} \left[\frac{\phi(DX) dz^1 \wedge \dots \wedge dz^n}{X^1 \dots X^n} \right] && \text{(by (13))} \\
&= (-1)^k \sum_i \frac{1}{\prod \alpha_i!} \cdot \frac{\partial^{|\alpha|} (\phi(DX) \det B)}{\partial z_1^{\alpha_1} \dots \partial z_n^{\alpha_n}} \Big|_{z=0} && \text{(by Lemma 2.1)}
\end{aligned}$$

4 Localization at a maximally degenerate zero on $\mathbb{C}P^n$

In this section we illustrate Theorem 1.3 by computing Futaki-Morita invariants for a holomorphic vector field on $\mathbb{C}P^n$ with a maximally degenerate zero. Proposition 4.1 in particular gives a localization formula for Chern numbers of $\mathbb{C}P^n$ without a summation over multiple points. As the maximally degenerate vector field we use is unique up to coordinate change, such a formula is essentially unique.

Let $A \in \mathfrak{sl}(n+1, \mathbb{C})$ be zero everywhere except for a diagonal of 1's above the main diagonal. A induces a holomorphic vector field $X = \sum A_{ij} Z_j \frac{\partial}{\partial Z_i}$ in homogeneous coordinates (we let the indices for A begin at 0 here). This vector field has a single zero at $p = [1, 0, \dots, 0]$, which is isolated and of maximal degeneracy. Changing to nonhomogeneous coordinates $z_i = Z_i/Z_0$ for $i = 1, \dots, n$ on $U_0 = \{Z_0 \neq 0\}$,

$$X = \sum_{j=1}^{n-1} (z_{j+1} - z_1 z_j) \frac{\partial}{\partial z_j} + (-z_1 z_n) \frac{\partial}{\partial z_n} \quad (15)$$

so that

$$DX|_{U_0} = \begin{bmatrix} -2z_1 & 1 & 0 & \dots & 0 \\ -z_2 & -z_1 & \ddots & \ddots & \vdots \\ \vdots & 0 & \ddots & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & 1 \\ -z_n & 0 & \dots & 0 & -z_1 \end{bmatrix} \quad (16)$$

In order to implement Theorem 1.3 we need to find B such that $z_i^{\alpha_i+1} = \sum B_{ij} X_j$.

To do this systematically choose $k \in \mathbb{Z}$ such that $2^k < n+1 \leq 2^{k+1}$. One may observe from (15)

$$\begin{aligned} z_1^{n+1} &= (-z_1)^{n-1}X_1 + (-z_1^{n-2})X_2 + \cdots + (-z_1)X_{n-1} + (-1)X_n \\ z_n^2 &= z_nX_{n-1} + (-z_{n-1})X_n \end{aligned}$$

while by completely factoring differences of squares in $z_{j+1}^{2^k} - (z_1z_j)^{2^k}$, we have for $j = 1, \dots, n-2$

$$z_{j+1}^{2^k} = \left(z_j^{2^k}\right) z_1^{2^k} + X_j \prod_{i=0}^{k-1} \left(z_{j+1}^{2^i} + z_1^{2^i} z_j^{2^i}\right)$$

By the choice of k and the above expression for z_1^{n+1} , these expressions recursively give $z_{j+1}^{2^k}$ as a linear combination of the X_j and thus contain the information necessary to form the desired matrix $B = (B_{ij})$ with $\alpha_1 = n, \alpha_n = 1$, and $\alpha_i = 2^k - 1$ for $i = 2, \dots, n-1$. It then follows from Theorem 1.3 that

$$\binom{n+k}{n} f_\phi(X) = (-1)^k \frac{1}{n![(2^k-1)!]^{n-2}} \frac{\partial(\phi(DX) \det B)}{(\partial z_1)^n (\partial z_2)^{2^k-1} \cdots (\partial z_{n-1})^{2^k-1} \partial z_n} \Big|_{z=0} \quad (17)$$

By using standard determinant properties, we find

$$\det B = (-1)^n (z_n + z_1 z_{n-1}) \prod_{j=1}^{n-2} \prod_{i=0}^{k-1} \left(z_{j+1}^{2^i} + z_1^{2^i} z_j^{2^i}\right)$$

Nearly all z_2, \dots, z_{n-1} derivatives of $\det B$ evaluated at $z_2 = \cdots = z_{n-1} = 0$ yield zero or a term with z_1^m where $m > n$, which may be ignored. The exception is when all $(2^k - 1)$ derivatives for each of z_2, \dots, z_{n-1} in (17) are applied to $\det B$, yielding

$$\frac{\partial \det B}{(\partial z_2)^{2^k-1} \cdots (\partial z_{n-1})^{2^k-1}} \Big|_{z_2=\cdots=z_{n-1}=0} = (-1)^n [(2^k-1)!]^{n-2} (z_1^n + z_n)$$

or when only one of these $(2^k - 1)^{n-2}$ derivatives is not applied, giving

$$\frac{\partial \det B}{(\partial z_2)^{2^k-1} \cdots (\partial z_j)^{2^k-2} \cdots (\partial z_{n-1})^{2^k-1}} \Big|_{z_2=\cdots=z_{n-1}=0} = (-1)^n \frac{[(2^k-1)!]^{n-2}}{2^k-1} z_1^j z_n$$

With these observations, (17) is evaluated to give

Proposition 4.1 *Let X be the maximally degenerate vector field on \mathbb{CP}^n given in (15). For any invariant polynomial ϕ of degree $n+k$, the Futaki-Morita integral is*

$$\binom{n+k}{n} f_\phi(X) = \frac{(-1)^{n+k}}{n!} \left(\frac{\partial^n \phi(DX)}{\partial z_1^n} + \sum_{j=2}^n \frac{\partial}{\partial z_1^n \partial z_j} (\phi(DX) \cdot z_1^j) \right) \Big|_{z=0}$$

where DX is as in (16).

A few simple cases of note:

- (i) Let $\phi(A) = \det(A)$, so $k = 0$ and $f_\phi(X)$ calculates the Euler characteristic $\chi(\mathbb{C}P^n)$. From (16),

$$\det(DX) = (-1)^n \left[2z_1^n + \sum_{j=2}^n z_j z_1^{n-j} \right]$$

Inserting this into Proposition 4.1 yields

$$\begin{aligned} \chi(\mathbb{C}P^n) &= \frac{(-1)^n}{n!} \left(\frac{\partial^n \det(DX)}{\partial z_1^n} + \sum_{j=2}^n \frac{\partial}{\partial z_1^n \partial z_j} (\det(DX) \cdot z_1^j) \right) \Big|_{z=0} \\ &= \frac{1}{n!} \left(2n! + \sum_{j=2}^n \frac{\partial}{\partial z_1^n} (z_1^j z_1^{n-j}) \right) \\ &= n + 1 \end{aligned}$$

- (ii) Take $\phi(A) = [\text{Tr}(A)]^n$. From (16),

$$\text{Tr}(DX) = -(n+1)z_1$$

By Proposition 4.1,

$$\begin{aligned} \int c_1^n &= \frac{(-1)^n}{n!} \frac{\partial}{\partial z_1^n} ((-(n+1)z_1)^n) \Big|_{z=0} \\ &= (n+1)^n \end{aligned}$$

One could at this point compute the entire cohomology ring of $\mathbb{C}P^n$ as in [13], but without the complicated summations.

- (iii) Take $\phi(A) = [\text{Tr}(A)]^{n+1}$, so that $f_\phi(X)$ calculates the Futaki invariant as in (8). By (16) again,

$$\phi(DX) = [-(n+1)z_1]^{n+1}.$$

It is immediate from Proposition 4.1 that $\text{Fut}(\mathbb{C}P^n, X) = 0$ as there are no derivatives of appropriate order. Of course this is necessary; the Fubini-Study metric on $\mathbb{C}P^n$ is well-known to be Kähler-Einstein.

- (iv) Similarly, one could check that $f_\phi(X)$ vanishes for $\phi(A) = \text{Tr}(A) \det A$ as there are again no derivatives of appropriate order. This vanishing was observed by Futaki to always be the case [11].

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